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Second-order generalized weak subdifferentials and applications to composite set-valued optimization problems

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ABSTRACT
In this paper, we firstly introduce a new second-order generalized weak subdifferential for a set-valued mapping by means of the second-order generalized Studniarski epiderivative and scalarization. We secondly discuss an existence theorem and some properties of the second-order generalized weak subdifferential. Finally, we obtain the optimality conditions for locally strict efficient solutions of order two in a composite set-valued optimization problem. Several main results improve and generalize the corresponding results in the literature.

KEYWORDS
Composite set-valued optimization problem, Optimality conditions, Second-order generalized weak subdifferential, Locally strict efficient solutions of order two

1. Introduction

The crucial significance of the subdifferential in optimization is widely acknowledged, and there are mainly two ways to define the notion of subdifferential. One approach is a generalization of the standard definition by using continuous linear mappings, see [1–7]. Another approach is based on a characterization of the subdifferential using different derivatives. In this framework, Clarke [8] defined the Clarke's subdifferential by means of the generalized directional derivative. Afterwards, the notions of subgradients and subdifferential have been investigated by many authors. Baier and Jahn [9] used contingent epiderivative to introduce the notion of subdifferential for a cone-convex set-valued mapping and discussed its properties. Song [10] introduced a notion of the weak contingent generalized gradient for set-valued mappings associated with the contingent epiderivative of set-valued mappings. Guo et al. [11] proved an existence theorem of the

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subdifferential for set-valued mappings introduced by Borwein [2] and gave relations between this subdifferential and the subdifferential introduced by Baier and Jahn [9]. Guo et al. [12] defined a generalized ϵ-subdifferential for a set-valued mapping by means of the radial epiderivative and a norm. They also gave a relationship between the existence of the radial epiderivative and the existence of the generalized ϵ-subdifferential for a set-valued mapping. Nguyen et al. [13] established KKT-type robust necessary conditions for robust (weakly) efficient solutions via the subdifferentials of $\mathcal{C}$-derivatives to such problems. Wei et al. [14] discussed Lagrange multiplier rules via Clarke subdifferentials for generalized optimization problems. Yalcin and Kasimbeyli [15] established relationships between the radial epiderivative, Clarke's directional derivative, Rockafellar's subderivative, and the directional derivative. They also presented explicit formulations for computing the radial epiderivatives in terms of weak subgradients and vice versa. By using tools from capacity theory, Alphonse and Wachsmuth [16] derived that derivatives of the solution maps of the penalised problems converge in the weak operator topology to an element of the strong-weak Bouligand subdifferential.

Recently, Anh [17] introduced second-order generalized subdifferential for a set-valued mapping by means of second-order generalized Studniarski epiderivative. Wang and Zhang [18] defined the second-order weak subdifferential of vector-valued mappings by means of scalarization and examined some properties of the concept. To the best of our knowledge, there are numerous set-valued mappings for which the above-mentioned second-order generalized subdifferentials [17] do not exist. To improve the case, motivated by the work in [12, 17, 18], we introduce a new second-order generalized weak subdifferential of set-valued mappings, establish some of its properties, and discuss its applications to optimality conditions of composite set-valued optimization.

This article is structured as follows. In Sect. 2, the primary concepts and definitions that will be used in the sequel are reviewed. In Sect. 3, a new second-order generalized weak subdifferential of set-valued mappings is introduced, and a few of its properties are investigated, such as convexity, the sum rule, etc. In Sect. 4, the second-order generalized weak subdifferential is exploited to set up the optimality conditions for locally strict efficient solutions of order two. In Sect. 5, a concise synopsis of the entire article is given.

2. Preliminaries

Throughout the paper, let $\mathbb{X}$ and $\mathbb{Y}$ be two real topological linear spaces, $\mathbb{R}^m$ be an m-dimensional Euclidean space, and $\gamma_1 \subseteq \mathbb{Y}$ be a solid closed convex pointed cone. $B_{\mathbb{X}}(x, r)$ is used for the closed ball with radius $r > 0$ and centered at $x \in \mathbb{X}$ and $\mathcal{U}(x)$ for the set of all neighborhoods of x. By $\mathbb{X}^*$ and $\mathbb{Y}^*$ we denote the topological dual spaces of $\mathbb{X}$ and $\mathbb{Y}$, respectively. The dual cone of γ_1 is defined as

$$\gamma_1^* = \{y_1^* \in \mathbb{Y}^* : \langle y_1^*, y_1 \rangle \geqslant 0, \quad \forall y_1 \in \gamma_1\}.$$

$0_{\mathbb{X}}$ and $0_{\mathbb{Y}}$ stand for the original points of $\mathbb{X}$ and $\mathbb{Y}$, respectively. The interior and closure of γ_1 are denoted by $\operatorname{int}\gamma_1$ and $\operatorname{cl}\gamma_1$, respectively. $\operatorname{cone}\mathscr{W}_1 = \{\lambda y_1 : \lambda \geq 0, \quad y_1 \in \mathscr{W}_1\}$ denotes the cone hull of $\mathscr{W}_1$, where $\mathscr{W}_1$ is a nonempty subset of $\mathbb{Y}$. The space $\mathbb{Y}$ is endowed with a partial order relation $\preceq_{\gamma_1}/\prec_{\gamma_1}$ induced by the cone γ_1 as follows: for any $y_1, y_2 \in \mathbb{Y}$

$$y_1 \preceq_{\gamma_1} y_2 \text{ if and only if } y_2 - y_1 \in \gamma_1,$$
$$y_1 \prec_{\gamma_1} y_2 \text{ if and only if } y_2 - y_1 \in \operatorname{int}\gamma_1.$$

Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping. The graph, epigraph and domain of $\aleph$ are, respectively, denoted by

$$\begin{aligned} \operatorname{gr} \aleph &= \{(x_1, y_1) \in \mathbb{X} \times \mathbb{Y} : x_1 \in \mathbb{X}, y_1 \in \aleph(x_1)\}, \\ \operatorname{epi} \aleph &= \{(x_1, y_1) \in \mathbb{X} \times \mathbb{Y} : x_1 \in \mathbb{X}, y_1 \in \aleph(x_1) + \gamma_1\} \end{aligned}$$

and

$$\operatorname{dom} \aleph = \{x_1 \in \mathbb{X} : \aleph(x_1) \neq \emptyset\}.$$

Let $y_0 \in \mathbb{Y}$. Denote

$$\aleph(\mathbb{X}) = \bigcup_{x \in \mathbb{X}} \aleph(x) \text{ and } \aleph(x) - y_0 = \aleph(x) - \{y_0\}.$$

In what follows we recall also some basic concepts and results.

Definition 2.1. [1] *Let $\mathscr{W}_1$ be a nonempty subset of $\mathbb{Y}$. $\mathscr{W}_1$ is said to be convex if for any $\delta \in [0,1]$ and for each $x_1, y_1 \in \mathscr{W}_1$,*

$$\delta x_1 + (1-\delta) y_1 \in \mathscr{W}_1.$$

Definition 2.2. [5] *A set-valued mapping $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ is said to be strictly positive homogeneous if*

$$\aleph(\delta x_1) = \delta \aleph(x_1), \quad \forall \delta > 0, x_1 \in \mathbb{X}.$$

Definition 2.3. [19, 20] *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$ and $\nu \in \aleph(\mu)$.*

(i) *A point (μ, ν) is said to be a minimizer of $\aleph$ on $\mathbb{X}'$, denoted by $(\mu, \nu) \in \operatorname{Min}_{\gamma_1} \aleph(\mathbb{X}')$ if*

$$(\aleph(\mathbb{X}') - \nu) \cap (-\gamma_1) = \{0_{\mathbb{Y}}\}.$$

(ii) *A point (μ, ν) is said to be a locally strict minimizer of order two, denoted by $(\mu, \nu) \in \operatorname{Strl}_{\gamma_1}(\aleph, \mathbb{X}'; 2)$ if there exist $\alpha > 0$ and $U \in \mathcal{U}(\mu)$ such that*

$$(\aleph(\mu) - \nu) \cap (-\gamma_1) = \{0_{\mathbb{Y}}\}$$

and

$$(\aleph(x) + \gamma_1) \cap B_{\mathbb{Y}}(\nu, \alpha \|x - \mu\|^2) = \emptyset, \quad \forall x \in (\mathbb{X}' \cap U) \setminus \{\mu\}.$$

Definition 2.4. [17, 21, 22] *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$ and $\nu \in \aleph(\mu)$.*

(i) *The contingent cone of $\mathbb{X}'$ at μ is defined by*

$$T_{\mathbb{X}'}(\mu) = \left\{x \in \mathbb{X} : \exists t_n \to 0^+, \exists x_n \to x, \text{ s.t. } \mu + t_n x_n \in \mathbb{X}'\right\}.$$

(ii) *The second-order Studniarski set of* $\operatorname{epi} \aleph$ *at (μ, ν) is defined by*

$$S^2_{\mathrm{epi}\,\aleph}(\mu,\nu) = \{(x,y) \in \mathbb{X} \times \mathbb{Y} : \exists t_n \to 0^+, \exists (x_n, y_n) \to (x,y) \\ s.t.\ \left(\mu + t_n x_n, \nu + t_n^2 y_n\right) \in \mathrm{epi}\,\aleph\}.$$

(iii) *The second-order generalized Studniarski epiderivative* $G\text{-}ED^2_S\aleph(\mu,\nu)$ *of* $\aleph$ *at* (μ,ν) *is the set-valued mapping from* $T_{\mathbb{X}'}(\mu)$ *to* $\mathbb{Y}$ *defined by*

$$G\text{-}ED^2_S\aleph(\mu,\nu)(x) = \mathrm{Min}_{\gamma_1}\left\{y \in \mathbb{Y} : (x,y) \in S^2_{\mathrm{epi}\,\aleph}(\mu,\nu)\right\}.$$

Remark 2.1. *From* [23, Proposition 2.3], *we know that* $T_{\mathbb{X}'}(\mu) = \mathrm{cl}\,\mathrm{cone}(\mathbb{X}' - \mu)$ *if* $\mathbb{X}'$ *is convex.*

Definition 2.5. [19, 24] *A subset* $\mathscr{W}_1$ *of* $\mathbb{Y}$ *fulfills the domination property if* $\mathscr{W}_1 \subseteq \mathrm{Min}_{\gamma_1}\mathscr{W}_1 + \gamma_1$.

Definition 2.6. [25] *A set-valued mapping* $\aleph : \mathcal{Q} \to 2^{\mathbb{Y}}$, *where* $\mathcal{Q}$ *is a nonempty set of* $\mathbb{X}$, *is called to be generalized* γ_1*-subconvexlike on* $\mathcal{Q}$ *if there exists* $\theta \in \mathrm{int}\,\gamma_1$ *such that for each* $x_1, x_2 \in \mathcal{Q}$, $\zeta \in [0,1]$ *and* $\epsilon > 0$, *there exist* $x_3 \in \mathcal{Q}$ *and* $\eta > 0$,

$$\epsilon\theta + \zeta\aleph(x_1) + (1-\zeta)\aleph(x_2) \subseteq \eta\aleph(x_3) + \gamma_1.$$

Remark 2.2. *It follows from* [25, Theorem 2.1] *that* $\mathrm{cone}\,\aleph(\mathcal{Q}) + \mathrm{int}\,\gamma_1$ *is convex if* $\aleph$ *is generalized* γ_1*-subconvexlike on* $\mathcal{Q}$.

3. Second-order generalized weak subdifferentials

In this section, we introduce a new second-order generalized weak subdifferential of set-valued mappings and discuss its existence and some properties. To start, we shall expound upon the lemma and definition below.

Lemma 3.1. *Let* $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ *be generalized* γ_1*-subconvexlike on* $\mathbb{X}$. *Then one and only one of the following statements is true:*

(i) *There exists* $x \in \mathbb{X}$ *such that*

$$\aleph(x) \cap (-\,\mathrm{int}\,\gamma_1) \neq \emptyset;$$

(ii) *There exists* $\varphi \in \gamma_1^* \backslash \{0_{\mathbb{Y}^*}\}$ *such that*

$$\langle \varphi, y \rangle \geq 0, \quad \forall y \in \aleph(\mathbb{X}).$$

Proof. By Remark 2.2, we obtain $\mathrm{cone}\,\aleph(\mathbb{X}) + \mathrm{int}\,\gamma_1$ is convex. The desired conclusion can be obtained by [26, Theorem 3.1]. Thus, the proof of this lemma is complete. □

Definition 3.1. [17] *Let* $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ *be a set-valued mapping,* $\mu \in \mathbb{X}' \subseteq \mathbb{X}$ *and* $\nu \in \aleph(\mu)$. *The second-order generalized subdifferential of* $\aleph$ *at* (μ,ν) *on* $\mathbb{X}'$ *is defined by*

$$\partial^2\aleph(\mu,\nu) = \left\{L \in \mathcal{L}(\mathbb{X},\mathbb{Y}) : G\text{-}ED^2_S\aleph(\mu,\nu)(x_1) \subseteq L(x_1) + \gamma_1, \quad \forall x_1 \in T_{\mathbb{X}'}(\mu)\right\},$$

where $\mathcal{L}(\mathbb{X},\mathbb{Y})$ *denotes the space of linear operators from* $\mathbb{X}$ *to* $\mathbb{Y}$.

Motivated by the subdifferentials in [12, 17, 18], we propose the new second-order generalized weak subdifferential for a set-valued mapping.

Definition 3.2. *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$, $\nu \in \aleph(\mu)$ and $G\text{-}ED^2_S\aleph(\mu,\nu)$ exists. A pair $(x_1^*, c, y_1^*) \in \mathbb{X}^* \times \mathbb{R}_+ \times \gamma_1^*$ is called the second-order generalized weak subgradient of $\aleph$ at (μ,ν) on $\mathbb{X}'$ if*

$$\langle y_1^*, y_1\rangle \geq \langle x_1^*, x_1\rangle - c\|x_1\|, \quad \forall x_1 \in T_{\mathbb{X}'}(\mu), y_1 \in G\text{-}ED^2_S\aleph(\mu,\nu)(x_1).$$

The set

$$\begin{aligned}&\partial^2_w\aleph(\mu,\nu)\\ &= \left\{(x_1^*, c, y_1^*) : \langle y_1^*, y_1\rangle \geq \langle x_1^*, x_1\rangle - c\|x_1\|, \quad \forall x_1 \in T_{\mathbb{X}'}(\mu), y_1 \in G\text{-}ED^2_S\aleph(\mu,\nu)(x_1)\right\}\end{aligned}$$

of all second-order generalized weak subgradients of $\aleph$ at (μ,ν) on $\mathbb{X}'$ is called the second-order generalized weak subdifferential of $\aleph$ at (μ,ν) on $\mathbb{X}'$. $\aleph$ is called second-order generalized weak subdifferentiable at (μ,ν) on $\mathbb{X}'$ if $\partial^2_w\aleph(\mu,\nu) \neq \emptyset$. Obviously, $(0_{\mathbb{X}^*}, 0, 0_{\mathbb{Y}^*}) \in \partial^2_w\aleph(\mu,\nu)$.

Remark 3.1. *It is clear that $\aleph$ is second-order generalized weak subdifferentiable at (μ,ν) if $\aleph$ is second-order generalized subdifferentiable at (μ,ν), and the converse may fail. So the notion of the second-order generalized weak subdifferential is better than the second-order generalized subdifferential.*

The next example illustrates Remark 3.1.

Example 3.1. *Let $\mathbb{X}' = \mathbb{X} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $\mathcal{L}(\mathbb{X},\mathbb{Y})$ be the space of linear operators from $\mathbb{X}$ into $\mathbb{Y}$, $\gamma_1 = \mathbb{R}^2_+$ and $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping given by $\aleph(x) = \{(y_1,y_2) \in \mathbb{Y} : y_1 + y_2 \geq x^2\}$. Take $(\mu,\nu) = (0,(0,0))$. Then we obtain*

$$G\text{-}ED^2_S\aleph(0,(0,0))(x) = \left\{(y_1,y_2) \in \mathbb{Y} : y_1 + y_2 = x^2\right\}$$

and

$$\begin{aligned}(0,0,(1,1)) \in\, &\partial^2_w\aleph(0,(0,0))\\ &= \left\{(x_1^*, c_1, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}^3_+ : x_1^* x \leq c_1|x| + y_1^* y_1 + y_2^* y_2, x^2 = y_1 + y_2\right\}.\end{aligned}$$

It follows from [17] that

$$\begin{aligned}&\partial^2\aleph(0,(0,0))\\ &= \left\{L \in \mathcal{L}(\mathbb{X},\mathbb{Y}) : G\text{-}ED^2_S\aleph(0,(0,0))(x) \subseteq L(x) + \gamma_1, \quad \forall x \in T_{\mathbb{X}'}(0)\right\} = \emptyset.\end{aligned}$$

We now discuss the existence of the second-order generalized weak subgradients.

Theorem 3.1. *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$, $\nu \in \aleph(\mu)$ and $G\text{-}ED^2_S\aleph(\mu,\nu)$ exists. If $G\text{-}ED^2_S\aleph(\mu,\nu)(T_{\mathbb{X}'}(\mu)) \cap (-\operatorname{int}\gamma_1) = \emptyset$ and $G\text{-}ED^2_S\aleph(\mu,\nu)$ is generalized γ_1-subconvexlike on $T_{\mathbb{X}'}(\mu)$, then $\partial^2_w\aleph(\mu,\nu) \neq \emptyset$.*

Proof. By Lemma 3.1, it follows that there exists $(x^*, c, y^*) \in \mathbb{X}^* \times \mathbb{R}_+ \times (\gamma_1^* \backslash \{0_{\mathbb{Y}^*}\})$

such that

$$\langle y^*, y\rangle \geq 0 = \langle x^*, x\rangle - c\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x).$$

Thus, $(0_{\mathbb{X}^*}, 0, y^*) \in \partial_w^2\aleph(\mu,\nu)$. This completes the proof. □

In the remaining sections, we always assume that second-order generalized Studniarski epiderivatives exist. Now we discuss some properties of the second-order generalized weak subdifferentials.

Proposition 3.1. *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$ and $\nu \in \aleph(\mu)$. Then the set $\partial_w^2\aleph(\mu,\nu)$ is convex.*

Proof. Let $(x_1, c_1, y_1), (x_2, c_2, y_2) \in \partial_w^2\aleph(\mu,\nu)$ and $\zeta \in [0,1]$. Then, it follows from Definition 3.2 that

$$\begin{aligned}\langle y_1, y\rangle &\geq \langle x_1, x\rangle - c_1\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x),\\ \langle y_2, y\rangle &\geq \langle x_2, x\rangle - c_2\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x)\end{aligned}$$

and

$$\zeta\langle y_1, y\rangle \geq \zeta\langle x_1, x\rangle - \zeta c_1\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x), \tag{1}$$

$$(1-\zeta)\langle y_2, y\rangle \geq (1-\zeta)\langle x_2, x\rangle - (1-\zeta)c_2\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x). \tag{2}$$

Together with (1) and (2), we have

$$\begin{aligned}\langle \zeta y_1 + (1-\zeta)y_2, y\rangle &\geq \langle \zeta x_1 + (1-\zeta)x_2, x\rangle\\ &\quad - (\zeta c_1 + (1-\zeta)c_2)\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2\aleph(\mu,\nu)(x)\end{aligned}$$

and

$$(\zeta x_1 + (1-\zeta)x_2, \zeta c_1 + (1-\zeta)c_2, \zeta y_1 + (1-\zeta)y_2) \in \partial_w^2\aleph(\mu,\nu).$$

Therefore, we obtain that the set $\partial_w^2\aleph(\mu,\nu)$ is convex. That's the end of the proof. □

Proposition 3.2. *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}$ and $\nu \in \aleph(\mu)$. If $\aleph$ is strictly positive homogeneous, then*

$$\partial_w^2\aleph(\delta\mu, \nu_1) = \partial_w^2\aleph(\mu,\nu), \quad \forall \delta > 0,$$

where $\nu_1 = \delta\nu$.

Proof. We first prove that $\partial_w^2\aleph(\delta\mu, \nu_1) \subseteq \partial_w^2\aleph(\mu,\nu)$.

Let $(x^*, c, y^*) \in \partial_w^2\aleph(\delta\mu, \nu_1)$. Note that, by Definition 3.2, we have

$$\langle y^*, y_1\rangle \geq \langle x^*, \delta x\rangle - c\|\delta x\|, \quad \forall \delta x \in T_{\mathbb{X}}(\delta\mu), y_1 \in G\text{-}ED_S^2\aleph(\delta\mu,\nu_1)(\delta x).$$

Since $\aleph$ is strictly positive homogeneous and $T_{\mathbb{X}}(\delta\mu) = T_{\mathbb{X}}(\mu)$, one gets

$$\begin{aligned} G\text{-}ED_S^2\aleph\,(\delta\mu,\nu_1)\,(\delta x) &= \mathrm{Min}_{\gamma_1}\left\{y_1 \in \mathbb{Y} : (\delta x, y_1) \in S^2_{\mathrm{epi}\,\aleph}\,(\delta\mu,\nu_1)\right\} \\ &= \mathrm{Min}_{\gamma_1}\left\{\delta y \in \mathbb{Y} : (x, y) \in S^2_{\mathrm{epi}\,\aleph}\,(\mu,\nu)\right\} \\ &= \delta\,\mathrm{Min}_{\gamma_1}\left\{y \in \mathbb{Y} : (x, y) \in S^2_{\mathrm{epi}\,\aleph}\,(\mu,\nu)\right\} \\ &= \delta G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x). \end{aligned}$$

Moreover,

$$\langle y^*, \delta y\rangle \geq \langle x^*, \delta x\rangle - c\|\delta x\|, \quad \forall x \in T_{\mathbb{X}}(\mu), y \in G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x),$$

i.e.,

$$\langle y^*, y\rangle \geq \langle x^*, x\rangle - c\|x\|, \quad \forall x \in T_{\mathbb{X}}(\mu), y \in G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x),$$

that is,

$$(x^*, c, y^*) \in \partial_w^2\aleph(\mu,\nu).$$

Thus, we obtain

$$\partial_w^2\aleph(\delta\mu,\nu_1) \subseteq \partial_w^2\aleph(\mu,\nu).$$

Next, we prove that $\partial_w^2\aleph(\mu,\nu) \subseteq \partial_w^2\aleph(\delta\mu,\nu_1)$.

Let $(x^*, c, y^*) \in \partial_w^2\aleph(\mu,\nu)$. Then, it follows from Definition 3.2 that

$$\langle y^*, y\rangle \geq \langle x^*, x\rangle - c\|x\|, \quad \forall x \in T_{\mathbb{X}}(\mu), y \in G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x).$$

Since $\delta > 0$, we have

$$\langle y^*, \delta y\rangle \geq \langle x^*, \delta x\rangle - c\|\delta x\|, \quad \forall x \in T_{\mathbb{X}}(\mu), y \in G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x).$$

Also, since $\aleph$ is strictly positive homogeneous and $T_{\mathbb{X}}(\delta\mu) = T_{\mathbb{X}}(\mu)$, it holds that

$$\delta G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x) = G\text{-}ED_S^2\aleph\,(\delta\mu,\nu_1)\,(\delta x).$$

Moreover,

$$\langle y^*, y_1\rangle \geq \langle x^*, \delta x\rangle - c\|\delta x\|, \quad \forall \delta x \in T_{\mathbb{X}}(\delta\mu), y_1 \in G\text{-}ED_S^2\aleph\,(\delta\mu,\nu_1)\,(\delta x),$$

i.e.,

$$(x^*, c, y^*) \in \partial_w^2\aleph(\delta\mu,\nu_1),$$

that is,

$$\partial_w^2\aleph(\mu,\nu) \subseteq \partial_w^2\aleph(\delta\mu,\nu_1).$$

Then, we obtain

$$\partial_w^2 \aleph(\delta\mu, \nu_1) = \partial_w^2 \aleph(\mu, \nu).$$

Thus, this completes the proof of the proposition. □

Proposition 3.3. *Let $\aleph : \mathbb{X} \to 2^{\mathbb{Y}}$ be a set-valued mapping, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$ and $\nu \in \aleph(\mu)$. If the following conditions are satisfied:*

(i) *$(x - \mu, y - \nu) \in S^2_{\mathrm{epi}\,\aleph}(\mu, \nu)$ for each $x \in \mathbb{X}'$ and $y \in \aleph(x)$;*

(ii) *$\left\{y \in \mathbb{Y} : (x, y) \in S^2_{\mathrm{epi}\,\aleph}(\mu, \nu)\right\}$ fulfills the domination property for each $x \in \mathbb{X}' - \mu$;*

then for each $(x_1^*, c, y_1^*) \in \partial_w^2 \aleph(\mu, \nu)$, $x_1 \in \mathbb{X}'$ and $y_1 \in \aleph(x)$,

$$\langle y_1^*, y_1 - \nu\rangle \in \langle x_1^*, x_1 - \mu\rangle - c\|x_1 - \mu\| + \mathbb{R}_+.$$

Proof. For each $x_1 \in \mathbb{X}'$ and $y_1 \in \aleph(x_1)$, it follows from the domination property that

$$y_1 - \nu \in G\text{-}ED^2_S \aleph(\mu, \nu)(x_1 - \mu) + \gamma_1.$$

From Definition 3.2, one gets

$$\langle y_1^*, y_1 - \nu\rangle \in \langle x_1^*, x_1 - \mu\rangle - c\|x_1 - \mu\| + \mathbb{R}_+.$$

The proof is complete. □

Proposition 3.4. *Let $\aleph$, G and $\aleph + G : \mathbb{X} \to 2^{\mathbb{Y}}$ be three set-valued mappings, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$, $\nu \in \aleph(\mu)$, $\nu_1 \in G(\mu)$ and $\nu_2 = \nu + \nu_1$. If $G\text{-}ED^2_S \aleph(\mu, \nu)(T_{\mathbb{X}'}(\mu)) \subseteq \gamma_1$, $G\text{-}ED^2_S G(\mu, \nu_1)(T_{\mathbb{X}'}(\mu)) \subseteq \gamma_1$ and $G\text{-}ED^2_S(\aleph + G)(\mu, \nu_2)(x) \subseteq G\text{-}ED^2_S \aleph(\mu, \nu)(x) + G\text{-}ED^2_S G(\mu, \nu_1)(x)$ for each $x \in T_{\mathbb{X}'}(\mu)$, then*

$$\partial_w^2 \aleph(\mu, \nu) + \partial_w^2 G(\mu, \nu_1) \subseteq \partial_w^2 (\aleph + G)(\mu, \nu_2).$$

Proof. Let $(x_1^*, c_1, y_1^*) \in \partial_w^2 \aleph(\mu, \nu)$ and $(x_2^*, c_2, y_2^*) \in \partial_w^2 G(\mu, \nu_1)$. Then, by Definition 3.2, we obtain

$$\langle y_1^*, y\rangle \geq \langle x_1^*, x\rangle - c_1\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED^2_S \aleph(\mu, \nu)(x)$$

and

$$\langle y_2^*, y_1\rangle \geq \langle x_2^*, x\rangle - c_2\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y_1 \in G\text{-}ED^2_S G(\mu, \nu_1)(x).$$

Since $G\text{-}ED^2_S \aleph(\mu, \nu)(T_{\mathbb{X}'}(\mu)) \subseteq \gamma_1$ and $G\text{-}ED^2_S G(\mu, \nu_1)(T_{\mathbb{X}'}(\mu)) \subseteq \gamma_1$, we have

$$\langle y_1^* + y_2^*, y\rangle \geq \langle x_1^*, x\rangle - c_1\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED^2_S \aleph(\mu, \nu)(x) \tag{3}$$

and

$$\langle y_1^* + y_2^*, y_1\rangle \geq \langle x_2^*, x\rangle - c_2\|x\|, \quad \forall x \in T_{\mathbb{X}'}(\mu), y_1 \in G\text{-}ED^2_S G(\mu, \nu_1)(x). \tag{4}$$

Together with (3) and (4), one has

$$\langle y_1^* + y_2^*, y + y_1 \rangle \geq \langle x_1^* + x_2^*, x \rangle - (c_1 + c_2)\|x\|,$$
$$\forall x \in T_{\mathbb{X}'}(\mu), y \in G\text{-}ED_S^2 \aleph(\mu, \nu)(x), y_1 \in G\text{-}ED_S^2 G(\mu, \nu_1)(x).$$

Since

$$G\text{-}ED_S^2(\aleph + G)(\mu, \nu_2)(x) \subseteq G\text{-}ED_S^2 \aleph(\mu, \nu)(x) + G\text{-}ED_S^2 G(\mu, \nu_1)(x), \quad \forall x \in T_{\mathbb{X}'}(\mu),$$

one gets

$$\langle y_1^* + y_2^*, y_2 \rangle \geq \langle x_1^* + x_2^*, x \rangle - (c_1 + c_2)\|x\|,$$
$$\forall x \in T_{\mathbb{X}'}(\mu), y_2 \in G\text{-}ED_S^2(\aleph + G)(\mu, \nu_2)(x).$$

So

$$(x_1^*, c_1, y_1^*) + (x_2^*, c_2, y_2^*) \in \partial_w^2(\aleph + G)(\mu, \nu_2).$$

Therefore,

$$\partial_w^2 \aleph(\mu, \nu) + \partial_w^2 G(\mu, \nu_1) \subseteq \partial_w^2(\aleph + G)(\mu, \nu_2).$$

The proof is complete. □

Remark 3.2. (i) *When* $G\text{-}ED_S^2 \aleph(\mu, \nu)(T_{\mathbb{X}'}(\mu)) \nsubseteq \gamma_1$, *the conclusion of Proposition 3.4 may not hold.*

(ii) $\sqrt{2}$ *can not be omitted in* [27, Theorem 4]. *However, Proposition 3.4 need not be pointed out.*

Now, we give two examples to illustrate Proposition 3.4 and Remark 3.2(i), respectively.

Example 3.2. *Let* $\mathbb{X}' = \mathbb{X} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $\gamma_1 = \mathbb{R}_+^2$ *and* $G : \mathbb{X} \to 2^{\mathbb{Y}}$ *be defined, respectively, by* $\aleph(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \geq x^2, y_2 \geq x^2\}$ *and* $G(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \geq 2x^2, y_2 \geq 2x^2\}$. *Take* $(\mu, \nu) = (\mu, \nu_1) = (\mu, \nu_2) = (0, (0, 0))$. *Then*

$$G\text{-}ED_S^2 \aleph(\mu, \nu)(x) = \{(x^2, x^2)\},$$

$$G\text{-}ED_S^2 G(\mu, \nu_1)(x) = \{(2x^2, 2x^2)\}$$

and

$$G\text{-}ED_S^2(\aleph + G)(\mu, \nu_2)(x) = \{(3x^2, 3x^2)\}.$$

By direct calculation, we get

$$\partial_w^2 \aleph(0, (0, 0)) = \left\{(x_1^*, c_1, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : |x_1^*| \leq c_1\right\},$$

$$\partial_w^2 G(0, (0, 0)) = \left\{(x_2^*, c_2, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : |x_2^*| \leq c_2\right\}$$

and

$$\partial_w^2(\aleph + G)(0,(0,0)) = \left\{(x^*, c, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : |x^*| \le c\right\}.$$

Obviously,

$$\begin{aligned}\partial_w^2\aleph(0,(0,0)) + \partial_w^2 G(0,(0,0)) &= \left\{(x^*, c, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : |x^*| \le c\right\} \\ &\subseteq \partial_w^2(\aleph + G)(0,(0,0)).\end{aligned}$$

So Proposition 3.4 holds here.

Example 3.3. *Let* $\mathbb{X}' = \mathbb{X} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $\gamma_1 = \mathbb{R}_+^2$ *and* $G : \mathbb{X} \to 2^{\mathbb{Y}}$ *be defined, respectively, by* $\aleph(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 + y_2 \ge x^2\}$ *and* $G(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \ge 2x^2, y_2 \ge 2x^2\}$. *Take* $(\mu, \nu) = (\mu, \nu_1) = (\mu, \nu_2) = (0,(0,0))$. *Then*

$$G\text{-}ED_S^2\aleph\,(\mu,\nu)\,(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 + y_2 = x^2\},$$

$$G\text{-}ED_S^2 G\,(\mu,\nu_1)\,(x) = \{2x^2, 2x^2\}$$

and

$$G\text{-}ED_S^2(\aleph + G)\,(\mu,\nu_2)\,(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 + y_2 = 5x^2\}.$$

By direct calculation, we get

$$\begin{aligned}(0,0,(1,1)) &\in \partial_w^2\aleph(0,(0,0)) \\ &= \left\{(x_1^*, c_1, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : x_1^* x \le c_1|x| + y_1^* y_1 + y_2^* y_2, x^2 = y_1 + y_2\right\},\end{aligned}$$

$$(0,0,(1,2)) \in \partial_w^2 G(0,(0,0)) = \left\{(x_2^*, c_2, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : |x_2^*| \le c_2\right\}$$

and

$$\begin{aligned}&\partial_w^2(\aleph + G)(0,(0,0)) \\ &= \left\{(x^*, c, (y_1^*, y_2^*)) \in \mathbb{R} \times \mathbb{R}_+^3 : x^* x \le c|x| + y_1^* y_1 + y_2^* y_2, 5x^2 = y_1 + y_2\right\}.\end{aligned}$$

Obviously,

$$\begin{aligned}(0,0,(1,1)) + (0,0,(1,2)) &= (0,0,(2,3)) \\ &\notin \partial_w^2(\aleph + G)(0,(0,0)).\end{aligned}$$

So one shows Remark 3.2(i) here.

By Proposition 3.4, we obtain the following corollary.

Corollary 3.1. *Let* $\aleph_i : \mathbb{X} \to 2^{\mathbb{Y}}$ *be set-valued mappings,* $i = 1, \ldots, l$, $\mu \in \mathbb{X}' \subseteq \mathbb{X}$, $\nu_i \in \aleph_i(\mu)$ *and* $\breve{z} = \sum_{i=1}^{l} \nu_i$. *If* $G\text{-}ED_S^2\aleph_i\,(\mu,\nu_i)\,(T_{\mathbb{X}'}(\mu)) \subseteq \gamma_1$, $i = 1, \ldots, l$ *and*

$G\text{-}ED^2_S(\sum_{i=1}^{l}\aleph_i)(\mu,\check{z})(x) \subseteq \sum_{i=1}^{l} G\text{-}ED^2_S\aleph_i(\mu,\nu_i)(x)$ for each $x \in T_{\mathbb{X}'}(\mu)$, then

$$\sum_{i=1}^{l}\partial^2_w\aleph_i(\mu,\nu_i) \subseteq \partial^2_w(\sum_{i=1}^{l}\aleph_i)(\mu,\check{z}).$$

4. Optimality conditions

In this section, let $\mathbb{Z}$ be a real topological linear space and $C \subseteq \mathbb{Z}$ be a closed convex pointed cone with $\operatorname{int} C \neq \emptyset$. We consider the following constrained set-valued optimization problem:

$$(\text{SOP})\begin{cases} \min\ \aleph(x), \\ \text{s.t. } G(x)\cap(-C) \neq \emptyset, \quad x \in \mathbb{X}', \end{cases}$$

where $\mathbb{X}'$ is a nonempty subset of $\mathbb{X}$, $\aleph : \mathbb{X}' \longrightarrow 2^{\mathbb{Y}}$ and $G : \mathbb{X}' \longrightarrow 2^{\mathbb{Z}}$ are two set-valued mappings. Let $A = \{x \in \mathbb{X}' : G(x)\cap(-C) \neq \emptyset\}$. Now, consider the following composite set-valued optimization problem for (SOP):

$$(\text{SOP}_\varphi)\begin{cases} \min\ \langle\varphi,\aleph(x)\rangle, \\ \text{s.t. } x \in A, \end{cases}$$

where $\varphi \in \mathbb{Y}^*\setminus\{0_{\mathbb{Y}^*}\}$.

Definition 4.1. [20] *Let $\mu \in A$ and $\nu \in \aleph(\mu)$. A point (μ,ν) is said to be a locally strict efficient solution of order two for* (SOP_φ) *if there exist $\alpha > 0$ and $U \in \mathcal{U}(\mu)$ such that*

$$(\langle\varphi,\aleph(\mu)\rangle - \langle\varphi,\nu\rangle)\cap(-\mathbb{R}_+) = \{0\}$$

and

$$(\langle\varphi,\aleph(x)\rangle + \mathbb{R}_+)\cap B_{\mathbb{R}}(\langle\varphi,\nu\rangle, \alpha\|x-\mu\|^2) = \emptyset, \quad \forall x \in (A\cap U)\setminus\{\mu\}.$$

The optimality conditions for locally strict efficient solutions of order two are given as follows.

Theorem 4.1. *Let $\aleph : A \to 2^{\mathbb{Y}}$, $\mu \in A$, $\nu \in \aleph(\mu)$, $(x-\mu, y-\nu) \in S^2_{\operatorname{epi}\aleph}(\mu,\nu)$ for each $x \in A$ and $y \in \aleph(x)$, $\left\{y \in \mathbb{Y} : (x,y) \in S^2_{\operatorname{epi}\aleph}(\mu,\nu)\right\}$ has the domination property for each $x \in A-\mu$, $(x^*,c,y_1^*) \in \partial^2_w\aleph(\mu,\nu)$ and $M = \{(x^*,c) \in \mathbb{X}^*\times\mathbb{R}_+ : (x^*,c,y_1^*) \in \partial^2_w\aleph(\mu,\nu)\}$.*

(i) *If (μ,ν) is a locally strict efficient solution of order two for* $(\text{SOP}_{y_1^*})$*, then there exist $U \in \mathcal{U}(\mu)$ and $\alpha > 0$ such that for each $(x_1^*,c_1) \in M$ and $x_1 \in (A\cap U)$, there exists $c_2 \in \mathbb{R}_+$,*

$$|\langle x_1^*, x_1-\mu\rangle - c_1\|x_1-\mu\|| - \alpha\|x_1-\mu\|^2 > -c_2.$$

(ii) *If there exist $U \in \mathcal{U}(\mu)$, $\alpha > 0$ and $(x_1^*,c_1) \in M$ such that for each $x_1 \in$*

$(A \cap U) \setminus \{\mu\}$, $c_2 \in \{\langle y_1^*, y_1 - \nu\rangle - \langle x_1^*, x_1 - \mu\rangle + c_1\|x_1 - \mu\| : y_1 \in \aleph(x_1)\}$ *and* $c_3 \in \mathbb{R}_+$,

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2 + c_3| - \alpha \|x_1 - \mu\|^2 > 0,$$

then (μ, ν) *is a locally strict efficient solution of order two for* $(\mathrm{SOP}_{y_1^*})$.

Proof. (i) According to Definition 4.1, there exist $U \in \mathcal{U}(\mu)$ and $\alpha > 0$ such that for each $x_1 \in (A \cap U) \setminus \{\mu\}$,

$$(\langle y_1^*, \aleph(x_1)\rangle - \langle y_1^*, \nu\rangle + \mathbb{R}_+) \cap B_{\mathbb{R}}\left(0, \alpha \|x_1 - \mu\|^2\right) = \emptyset.$$

Thus, for each $y_1 \in \aleph(x_1)$,

$$|\langle y_1^*, y_1 - \nu\rangle| > \alpha \|x_1 - \mu\|^2. \tag{5}$$

By Proposition 3.3, we have for each $(x_1^*, c_1) \in M$, there exists $c_2 \in \mathbb{R}_+$ such that

$$\langle y_1^*, y_1 - \nu\rangle = \langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2. \tag{6}$$

It follows from (5) and (6) that

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2| > \alpha \|x_1 - \mu\|^2.$$

Therefore,

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\|| - \alpha \|x_1 - \mu\|^2 > -c_2.$$

(ii) Suppose to the contrary that (μ, ν) is not a locally strict efficient solution of order two for $(\mathrm{SOP}_{y_1^*})$. Next, we consider two cases.

Case 1: there exists $y \in \aleph(\mu)$ such that

$$(\langle y_1^*, \aleph(\mu)\rangle - \langle y_1^*, \nu\rangle) \cap (-\operatorname{int} \mathbb{R}_+) \neq \emptyset. \tag{7}$$

According to Proposition 3.3, one has

$$\langle y_1^*, \aleph(\mu) - \nu\rangle \subseteq \mathbb{R}_+,$$

which contradicts (7).

Case 2: for each $U \in \mathcal{U}(\mu)$ and $\alpha > 0$, there exists $x_1 \in (A \cap U) \setminus \{\mu\}$ such that

$$(\langle y_1^*, \aleph(x_1)\rangle - \langle y_1^*, \nu\rangle + \mathbb{R}_+) \cap B_{\mathbb{R}}\left(0, \alpha \|x_1 - \mu\|^2\right) \neq \emptyset.$$

Thus, there exist $y_1 \in \aleph(x_1)$ and $c_3 \in \mathbb{R}_+$ such that

$$|\langle y_1^*, y_1 - \nu\rangle + c_3| \leq \alpha \|x_1 - \mu\|^2. \tag{8}$$

By Proposition 3.3, there exists $c_2 \in \mathbb{R}_+$ such that

$$\langle y_1^*, y_1 - \nu\rangle = \langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2. \tag{9}$$

It follows from (8) and (9) that

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2 + c_3| \le \alpha \|x_1 - \mu\|^2,$$

which contradicts the assumption. The proof is complete. $\square$

Remark 4.1. *Since Theorem 4.1 does not involve the assumption of mth-order γ_1-convexity, it improves and generalizes* [17, Theorem 3.1].

Now, we give the next example to illustrate Theorem 4.1 and Remark 4.1.

Example 4.1. *Let $\mathbb{X}' = \mathbb{X} = \mathbb{R}$, $\mathbb{Y} = \mathbb{Z} = \mathbb{R}^2$, $\gamma_1 = C = \mathbb{R}^2_+$, $\aleph : \mathbb{X}' \to 2^{\mathbb{Y}}$ be a set-valued mapping given by $\aleph(x) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \ge 2x^2, y_2 \ge x^2\}$ and $G : \mathbb{X}' \to 2^{\mathbb{Z}}$ be a set-valued mapping defined by $G(x) = \{(y_1, y_2) \in \mathbb{Z} : x - 2 \le y_1, x - 2 \le y_2\}$. Then $A = \{x \in \mathbb{X} : x \le 2\}$. Take $(\mu, \nu) = (0, (0, 0))$, $x_1^* = 0$, $c_1 = 0$, $c_2 = 2$ $y_1^* = (1, 1)$, $\alpha = 1$ and $U = (-1, 1)$.*

Calculations give that

$$S^2_{\mathrm{epi}\,\aleph}(\mu, \nu) = \{(x, (y_1, y_2)) \in \mathbb{X} \times \mathbb{Y} : y_1 \ge 2x^2, y_2 \ge x^2\},$$

$$G\text{-}ED^2_S\aleph(\mu, \nu)(x) = \{(2x^2, x^2)\},$$

$$(0, 0, (1, 1)) \in \partial^2_w \aleph(\mu, \nu)$$

and

$$M = \{(x^*, c) \in \mathbb{R} \times \mathbb{R}_+ : |x^*| \le c\}.$$

It is obvious that (μ, ν) is a locally strict efficient solution of order two for $(\mathrm{SOP}_{y_1^*})$ *and*

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\|| - \alpha \|x_1 - \mu\|^2 > -c_2, \quad \forall x_1 \in (A \cap U).$$

So one shows Theorem 4.1(i) here.

Set $x_1 = \frac{3}{4}$, $y_1 = (\frac{9}{8}, \frac{9}{16})$, $c_2 = \frac{27}{16}$. Then, for each $c_3 \in \mathbb{R}_+$, we have

$$|\langle x_1^*, x_1 - \mu\rangle - c_1\|x_1 - \mu\| + c_2 + c_3| - \alpha \|x_1 - \mu\|^2 > 0,$$

one shows Theorem 4.1(ii) holds here.

Take $m = 2$, $x_2 = 2$, $x_3 = 1$ and $\zeta = \frac{1}{2}$, we can check that

$$\zeta^2\aleph(x_2) + (1 - \zeta^2)\aleph(x_3) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \ge \frac{7}{2}, y_2 \ge \frac{7}{4}\}$$

and

$$\aleph(\zeta x_2 + (1 - \zeta)x_3) = \{(y_1, y_2) \in \mathbb{Y} : y_1 \ge \frac{9}{2}, y_2 \ge \frac{9}{4}\}.$$

It is obvious that $\zeta^2\aleph(x_2) + (1 - \zeta^2)\aleph(x_3) \nsubseteq \aleph(\zeta x_2 + (1 - \zeta)x_2) + \gamma_1$. In consequence, $\aleph$ is not a second-order γ_1-convex set-valued mapping on A. So Remark 4.1 holds here.

5. Conclusion

In this article, the locally strict efficient solutions of order two for the composite set-valued optimization problems are studied. A novel second-order generalized weak subdifferential of set-valued mappings is not only founded, but also some of its properties are investigated. Finally, the optimality conditions for locally strict efficient solutions of order two without any convexity are constructed.

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